

### 97.16 Probability analysis for rolls of a square cuboidal die

Consider a six-sided die in which two opposite faces are squares of sides  $y$  and the other four faces are rectangles of sides  $x$  and  $y$ , as sketched in Figure 1. One can think of it as an ordinary cubical die that has been stretched or compressed along one axis. There is some probability  $p$  that one of the two square faces will show uppermost after such a die is randomly tossed, and a probability  $1 - p$  that one of the four rectangular faces will show. For example, in the limit that  $y$  becomes equal to  $x$  so that the die is cubical, then the probability that the original square faces come up is  $p = \frac{1}{3}$ . Over 30 years ago, the question was asked [1] what is  $p$  if  $y$  is not equal to  $x$ ? The author of that article cut 15 such dice out of a steel bar and had his students roll each of them  $N$  times to experimentally determine  $p$ . His results are listed in Table 1. A simple explanation was subsequently hypothesized in which  $p$  is proportional to the solid angle subtended by a square face [2], but it does not match the data well. In the present paper, an alternative simple model is proposed that better fits the measurements.

$x/y$	$p$	$N$
0.47	0.916	332
0.63	0.738	840
0.75	0.548	799
0.81	0.496	740
0.93	0.399	516
0.97	0.385	530
1.16	0.148	1011
1.23	0.154	532
1.44	0.052	654
1.55	0.040	606
1.60	0.017	702
1.71	0.031	609
1.87	0.009	680
2.12	0.007	275
2.65	0.006	503

TABLE 1

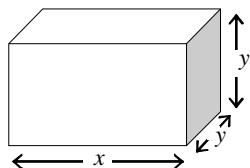


FIGURE 1

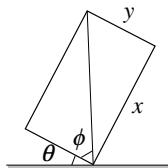


FIGURE 2

The model is developed in three steps. First suppose the die lands vertically without initial rotation on any one of its eight edges of length  $y$  in

Figure 1. Let the bottom square face make angle  $\theta$  with respect to the floor at the instant of landing, as sketched in Figure 2. The die will subsequently fall over either onto that bottom face or onto the right-hand rectangular face, depending on whether the centre of mass of the cuboid is to the left or the right of the edge that initially hits the floor [3]. In other words, it depends on whether a diagonal drawn across the front face of the die, as shown in Figure 2, is inclined leftward or rightward of vertical. That diagonal makes angle  $\phi = \tan^{-1}(x/y)$  relative to the bottom face and it will be exactly vertical when the die makes angle  $\theta_0 = \pi/2 - \phi$  relative to the floor. The die will fall onto the square face if  $0 < \theta < \theta_0$ . Assuming that all landing angles  $\theta$  between 0 and  $\pi/2$  are equally likely, the probability  $p_{2D}$  that the die ends with the square face up in this two-dimensional case without initial rotation is therefore

$$p_{2D} = \frac{\theta_0}{\pi/2} = 1 - \frac{2}{\pi} \tan^{-1}\left(\frac{x}{y}\right).$$

Intuitively it makes sense that the probability only depends on the ratio of  $x$  and  $y$ , and not on those two lengths independently.

The second step is to extend this result to find the probability  $p_{3D}$  of ending with a square face up if the die lands vertically without initial rotation on any of its twelve edges as the cuboid is reoriented in three dimensions. The much more likely case that it lands on a corner can be thought of as a superposition of landing on the two nearest lower edges of the die and is therefore included in  $p_{3D}$ . (If the corner of a large box is held against the floor and the box is then released, separate rotations about each of the lower edges are observed as the box settles.) If the die in Figure 1 lands on one of the four edges of length  $x$ , it will end with a rectangular face up. So  $p_{3D}$  can be calculated by weighting  $p_{2D}$  by the fractional probability  $8y/(8y + 4x)$  that the die (which spins about a *random* axis while in the air) will land on an edge of length  $y$  instead of one of length  $x$ ,

$$p_{3D} = \frac{8y}{8y + 4x} p_{2D} = \frac{2}{2 + x/y} \left[ 1 - \frac{2}{\pi} \tan^{-1}\left(\frac{x}{y}\right) \right]. \quad (1)$$

This equation correctly predicts  $p_{3D} = \frac{1}{3}$  for a cubical die with  $x = y$ .

The final step is to incorporate the fact that when a real die is thrown, it lands on the table with substantial initial horizontal translational and angular velocities. Consequently it tumbles along the table for a while. The die slows down each time a face inelastically hits the table [4]. It can be supposed that the loss is proportional to the area  $A$  of the colliding face (and hence the die is even more likely to end its motion on a larger area face) for two reasons. First, a greater number of elastic modes of vibration can be excited over a bigger surface. (As proof, a block of wood with an aspect ratio  $x/y$  of about 10 was dropped onto a concrete floor from a height of about 1 m. It bounced noticeably higher when dropped on its end compared to on its side.) Secondly, the larger faces of the cuboid will hit the table with higher impact speed (and hence loss) than the smaller faces as a die rolls around any edge  $y$  because the die has lower centre of mass in the former orientations. To model this rolling

loss, the factors of  $x/y$  in (1) can thus be weighted by some positive power of the ratio of the rectangular and square areas,  $xy/y^2$ , to give a final prediction of

$$p = \frac{1}{2 + (x/y)^n} \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{x}{y} \right)^n \right] \quad (2)$$

where  $n > 1$ . The exponent  $n$  effectively makes a thin die (for which  $x < y$ ) thinner and a thick die ( $x > y$ ) thicker. For example, if  $n = 3$  then a rolled die whose rectangular edge  $x$  is half the length of its square edge  $y$  will behave like a die with  $x = y/8$  that lands vertically without initial rotation. This example suggests that the value of  $n$  may depend on the height and flick of the wrist used to throw the die. One might also expect  $n$  to depend on the coefficient of restitution (that is, on the hardness of the die and table), on any slight beveling of the edges of the die, and on similar factors. In support of this expectation, a second set of data for square cuboidal dice made of polyvinylchloride, rather than steel, has been collected by Heilbronner [5] as listed in Table 2; they have somewhat different values of  $p$ , presumably because of their different compositions. It is evidently necessary that any general model such as (2) have some empirical parameter in it. For example, another way to account for the reduced likelihood of the die ending up in an orientation with high centre of mass is to weight the probability by a Boltzmann exponential  $\exp(-\beta E)$ , where  $E$  is the gravitational potential energy of the centre of mass of the die and  $\beta$  is taken to be a fit parameter [5, 6, 7].

$x/y$	$p$	$N$
0.2	0.974	2145
0.4	0.883	2184
0.6	0.741	2103
0.8	0.556	2238
1.0	0.333	—
1.2	0.191	2202
1.4	0.106	2259
1.6	0.072	2250

TABLE 2

The value of the parameter  $n$  is found by fitting (2) to the data. The results are graphed in Figure 3 for two values of  $n$  that span the range of variation of both Budden's and Heilbronner's experimental points. The conclusion is that  $n = 3.0 \pm 0.5$  for cuboidal dice rolled in an ordinary manner. (Consequently even a slight shaving of one face of a die would significantly alter its fairness.) This same model could be applied to the related problem of the probability  $1 - p$  for a tossed coin of thickness  $x$  and diameter  $y$  to land on edge [2, 4, 8, 9] if one approximates a coin as behaving like the smallest cuboid that would enclose it. In particular, (2) predicts the probability is between 0.12% and 0.83% for a British one-pound coin (of 3.15 mm thickness and 22.5 mm diameter) if  $n$  is between 2.5 and 3.5; this range brackets the experimental observation of 6 edge landings out of 1000 tosses [8].

—  $n = 2.5$  —  $n = 3.5$  ● Budden experiment ■ Heilbronner experiment

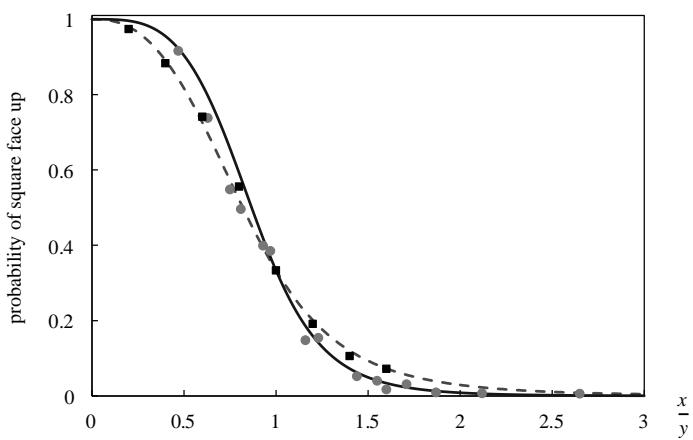


FIGURE 3

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